Tutorial for Final exam Phys 356, April 7th, 2006, Rm 126, 2:30 p.m.

I strongly recommend starting this test without the book under the same conditions as the Final exam (No books, no notes, no calculators, formula sheet provided)! You have an opportunity to test where your problems are and how you will be doing in the final exam.

Solve all problems.

1. A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and its dielectric constant is ε_r.

2. What current density would produce the vector potential, $\vec{A} = k\,\vec{u}_\Phi$ (where k is a constant), in cylindrical coordinates?

3. A point charge q is embedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R).
   a. Find the electric fields $\vec{D}$ and $\vec{E}$, the polarization $\vec{P}$, inside and outside the sphere.
   b. Find the bound charge densities, ρ_b and σ_b.
   c. What is the total bound charge on the surface?
   d. Where is the compensating negative bound charge located?

4. Calculate the energy stored in a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), which carries a total of N turns.

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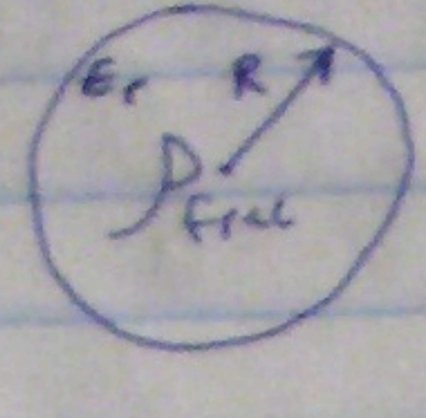

$\underline{r<R}$

$$\oint \vec{D}\cdot d\vec{a} = Q_{free\ enc.}$$

$$D(\cancel{4\pi r^2}) = \rho\left(\frac{4}{3}\cancel{\pi} r^{\cancel{3}}\right) \qquad \vec{D} = \frac{\rho r}{3}\hat{r}$$

For linear media:

$$\vec{D} = \epsilon\vec{E} \qquad \vec{E} = \frac{\vec{D}}{\epsilon_0\epsilon_r} \qquad \vec{E} = \frac{\rho r}{3\epsilon_0\epsilon_r}\hat{r} \text{ for } r<R$$

$$\vec{D} = \epsilon_0\epsilon_r\vec{E}$$

$\underline{r>R}$

$$\oint \vec{D}\cdot d\vec{a} = Q_{fenc}$$

$$D(\cancel{4\pi} r^2) = \rho\left(\frac{4}{3}\cancel{\pi} R^3\right) \qquad \vec{D} = \frac{\rho R^3}{3r^2}\hat{r}$$

$$\vec{E} = \frac{\vec{D}}{\epsilon_0\epsilon_r} = \frac{\vec{D}}{\epsilon_0\frac{\epsilon}{\epsilon_0}} = \frac{\vec{D}}{\epsilon_0(1)}$$

$$\vec{E} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{r} \quad \text{for } r>R$$

$\underline{\text{Potential}}$

$$V = -\int_\infty^0 \vec{E}\cdot d\vec{l} = -\int_\infty^R \frac{\rho R^3}{3\epsilon_0 r^2}dr - \int_R^0 \frac{\rho r}{3\epsilon_0\epsilon_r}dr$$

$$= \frac{\rho R^3}{3\epsilon_0}\frac{1}{r}\Big|_\infty^R - \frac{\rho}{6\epsilon_0\epsilon_r}r^2\Big|_R^0$$

$$= \frac{\rho R^2}{3\epsilon_0} - 0 - 0 + \frac{\rho R^2}{6\epsilon_0\epsilon_r} = \boxed{\frac{\rho R^2}{3\epsilon_0}\left[1+\frac{1}{2\epsilon_r}\right]}$$

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(2) $\vec{A} = k\,\vec{u}_\phi$ $\quad$ $k = \text{const}$ $\quad$ in cylindrical coord.

$\vec{B} = \nabla \times \vec{A}$ and $\nabla \times \vec{B} = \mu_0 \vec{J}$

so:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{s}\left[\frac{\partial}{\partial s}\left[s(k)\right]\right]\vec{a}_z$$

$$= \frac{k}{s}\,\vec{u}_z$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = -\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\vec{u}_\phi = +\frac{k}{s^2}\,\vec{u}_\phi$$

Thusly, since

$$\mu_0 \vec{J} = \frac{k}{s^2}\,\vec{u}_\phi$$

$$\boxed{\vec{J} = \frac{k}{\mu_0 s^2}\,\vec{u}_\phi}$$

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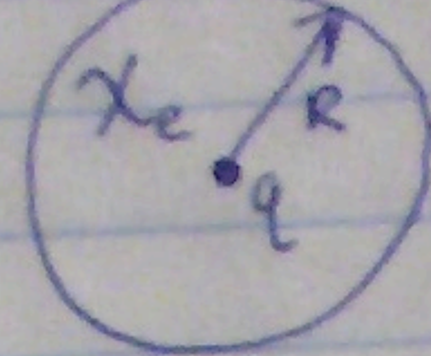
③ 

a) $\vec{D}$, $\vec{E}$ and $\vec{P}$ inside and outside sphere

$$\oint \vec{D}\cdot d\vec{a} = Q_{fenc} = q \qquad D(4\pi r^2) = q$$

$$\vec{D} = \frac{q}{4\pi r^2}\hat{r} \quad (\text{all } r)$$

inside, $r < R$

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon_0(1+\chi_e)\vec{E} \qquad \underline{\underline{\vec{E}}} = \frac{\vec{D}}{\epsilon_0(1+\chi_e)} = \boxed{\frac{q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{r}}$$

(linear media)

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \qquad \underline{\underline{\vec{P}}} = \frac{q(\epsilon_0\chi_e)}{4\pi\cancel{\epsilon_0}(1+\chi_e)r^2}\hat{r} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)r^2}\hat{r}}$$

outside, $r > R$

right away, $\vec{P} = \vec{0}$ (no material)

$$\vec{D} = \epsilon_0\epsilon_r\vec{E} \qquad \vec{E} = \frac{\vec{D}}{\epsilon_0\epsilon_r} = \frac{\vec{D}}{\epsilon_0} = \boxed{\frac{q}{4\pi r^2\epsilon_0}\hat{r}}$$

b) ρ_B and σ_B $\qquad \rho_B = -\vec{\nabla}\cdot\vec{P} \qquad \sigma_B = \vec{P}\cdot\hat{n}$

$$\underline{\underline{\rho_B}} = -\vec{\nabla}\cdot P = \frac{-q\chi_e}{4\pi\cdot(1+\chi_e)}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{-q\chi_e}{\cancel{4\pi}\cdot(1+\chi_e)}\left[\cancel{4\pi}\delta^3(\vec{r})\right] = \boxed{\frac{-q\chi_e\delta^3(\vec{r})}{(1+\chi_e)}}$$

$$\underline{\underline{\sigma_B}} = \vec{P}\cdot\hat{n} = \frac{q\chi_e}{4\pi(1+\chi_e)r^2}\left(\overset{1}{\cancel{\hat{r}\cdot\hat{r}}}\right) = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)r^2}}$$

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3c) $\underline{\underline{Q_{B\,\text{surface}}}} = \int \sigma_B \cdot da = \sigma_B A = \sigma_B (4\pi R^2)$

$$= \frac{q\,\chi_e}{\cancel{4\pi r^2}(1+\chi_e)} \cancel{(4\pi r^2)} = \boxed{\frac{q\,\chi_e}{1+\chi_e}}$$

d) The compensating negative bound charge must be at the center of the sphere.

$$\underline{\underline{q_B}} = \iiint \rho_B \, d\tau$$

$$= \iiint \frac{-q\,\chi_e}{(1+\chi_e)} \delta^3(\vec{r}) \, d\tau$$

$$= \boxed{\frac{-q\,\chi_e}{1+\chi_e}}$$ at the center

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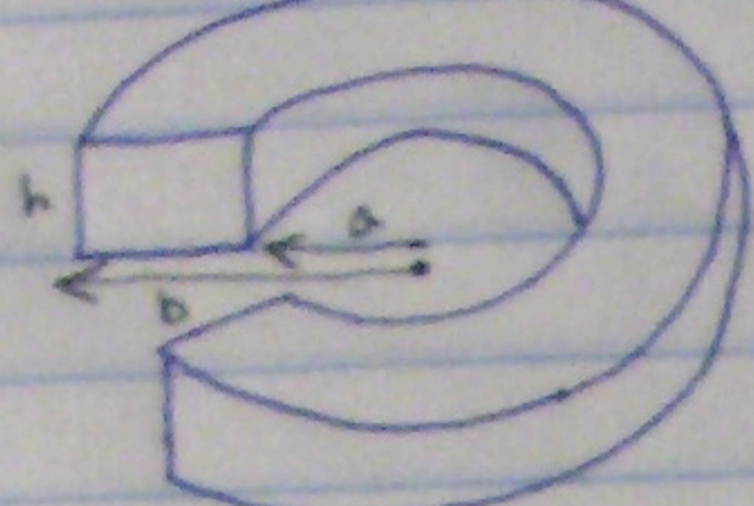
N turns

④ Energy stored in a toroidal coil.



We know that $\vec{B}_{toroid} = \begin{cases} 0 & \text{outside the toroid} \\ \dfrac{\mu_0 N I}{2\pi s}\,\vec{u}_\phi & \text{inside the toroid} \end{cases}$

$$W = \frac{1}{2\mu_0}\iiint B^2\, d\tau$$

so employ cylindrical coordinates:

$$\underline{\underline{W}} = \frac{1}{2\mu_0}\iiint \left[\frac{\mu_0 N I}{2\pi s}\right]^2 d\tau$$

$$= \frac{\mu_0 N^2 I^2}{8\pi^2}\int_0^h\int_0^{2\pi}\int_a^b \frac{1}{s^2}\, s\, ds\, d\phi\, dz$$

$$= \frac{\mu_0 N^2 I^2}{8\pi^2}(2\pi h)(\ln b - \ln a)$$

$$= \boxed{\frac{\mu_0 N^2 I^2 h}{4\pi}\ln\left(\frac{b}{a}\right)}$$

Note: there will be a fifth question related to theory, similar to the midterm.

Good luck everyone. Have a great summer.
>> Paulsen

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